Indian Statistical Institute, Bangalore

B. Math. Second Year

First Semester - Analysis III

Back Paper Exam Date : Jan 06, 2015

All questions carry equal marks, total marks: 50

1. (a) Let (f_n) be a sequence of functions on a metric space E that converges uniformly to a function f on E. Suppose each f_n is continuous. Show that f is continuous on E *(Marks: 4)*.

   (b) Let (f_n) be a sequence of Riemann-integrable functions on $[a, b]$ that converges uniformly to a function f on $[a, b]$. Prove that $f \in \mathcal{R}[a, b]$ and $\int f_n \to \int f$.

2. (a) Let (f_n) be a sequence of increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$ for all $x \in \mathbb{R}$ and all n. Prove that (f_n) has a subsequence that converges pointwise.

   (b) Prove that every uniformly convergent sequence of bounded functions is uniformly bounded *(Marks: 3)*.

3. (a) Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Find a necessary and sufficient condition for f to have potential (Marks: 5).

   (b) If f is a continuous function on an open subset S of $\mathbb{R}^n$ such that $\int f d\alpha = 0$ for any curve α in S, show that $f = 0$.

4. (a) Let P and Q be scalar valued continuously differentiable functions on $R = [a, b] \times [c, d]$ and α be the (positively oriented) boundary C of R. Prove that $\int \int_R (\partial Q/\partial x - \partial P/\partial y) dx dy = \int_C P dx + Q dy$.

   (b) Suppose S is a surface whose boundary is $x^2 + y^2 = 1$ and $f(x, y, z) = (xe^{z^3 - 2z^2}, \sin(xyz^2) + y, e^z \sin z^2 \cos(xy))$. Find $\int \int_S \mathrm{curl} f \cdot n dS$ *(Marks: 4)*.

5. (a) Let $F: \mathbb{R}^d \to \mathbb{R}^d$ be a continuously differentiable map. If $F(0) = 0$ and $F'(0)$ is invertible, prove that there is a neighborhood V of 0 in $\mathbb{R}^d$ and primitive maps $G_1, \cdots, G_d$ on V such that $F = F'(0) G_1 \cdots G_d$ on V.

   (b) Give an example of an infinitely differentiable function ϕ on $\mathbb{R}^d$ such that $S(\phi) \subset B_r(0)$ but $\phi = 1$ on $B_s(0)$ where $0 < s < r$ *(Marks: 4)*.